Mathematics in Civilization

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1.1 Numeral Systems and Counting in the Past

1.1.1 Hindu-Arabic System and Early Positional Systems

Counting in the early era

- Systems used for showing numbers are called "<u>numeral systems</u>".
- We found an numerical evidence from about thirty thousand years ago, showing number recorded in the early era by using "counting rod".

1.1.1 Hindu-Arabic System and Early Positional Systems

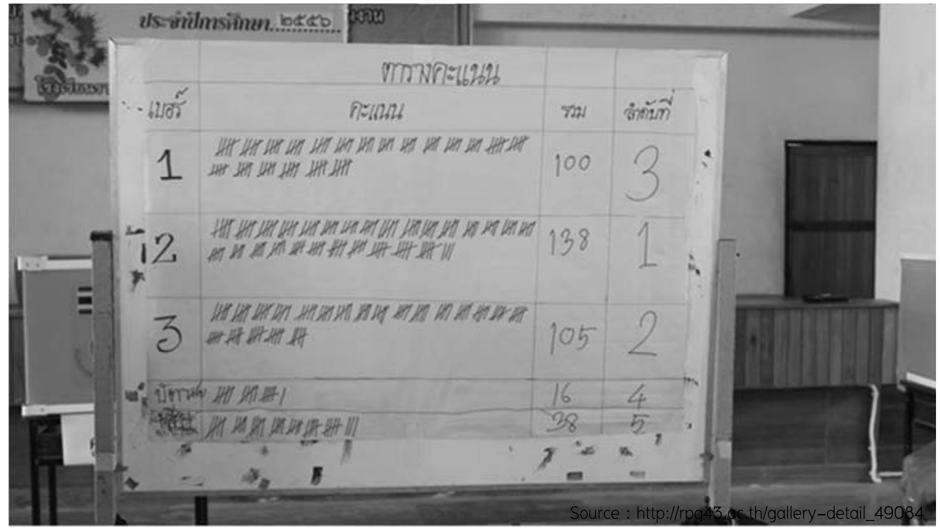
- The wooden rods are used as scores showing the quantity in Africa and Eastern Europe.
- We found notch on a bone for number recording.

An ishango bone was found in the Congo with two identical markings of sixty scratches each and equally numbered groups on the back.

Source : vedicsciences.net

1.1.1 Hindu-Arabic System and Early Positional Systems

- We found that the counting rods (vertical lines) were used on cave walls for counting animals.
- "Counting rods" had also been used in British taxation from the 13th century to the 1820s. After that people began to record on paper instead.



Source : http://rpg43.ac.th/gallery-detail_49084

- In the present, we still use rods for counting, such as counting election scores, counting competition scores, and recordings statistical data.

1.1.1 Hindu-Arabic System and Early Positional Systems

- This is how to use counting sticks associated with counting up to five. It is similar to a slat door.

Slat door counting system

- In South America, a five-line system has also been used in a different way.

Counting system of South Americans

1.1.1 Hindu-Arabic System and Early Positional Systems

- Human beings have developed symbols to represent numbers. Each method may exhibit the same number of different variations.

	IX	9
scratching	Roman numeral system	Hindu-Arabic system

1.1.1 Hindu-Arabic System and Early Positional Systems

- Each system has basic numbers and a different numbering rule.
- We now use the Hindu-Arabic numberal system invented in

India → Arab → Europe

- Today computers use Hindu-Arabic numeral system.

Counting Words

- At the early era, humans did not need to count much. If we look at the spoken language, we see words for "<u>one</u>" and "<u>two</u>". However, when we look for a "<u>three</u>", it is usually translated as "<u>many</u>". Maybe they didn't want to count up to 3.
- In Tasmania in Australia, there is a counting system we can translate to be "**<u>One</u>**" "**<u>Two</u>**" and "**<u>Many</u>**".
- In Queensland in Australia, there is counting record we can translate to be

"**<u>One</u>**" "**<u>Two</u>**" "**<u>One and Two</u>**" "**<u>Two and Two</u>**" and "**<u>Many</u>.**"

Counting Words

- In many nations, people name days in the past and the future not more than 3 days from the present day. For example, in Thai language,
  - we call one day or two days in the past that "เมื่อวาน" or " Meu Wan" (yesterday) และ "เมื่อวานซืน" or "Meu Wan Suen" (the day before yesterday), respectively.

Counting Words

- We call one day and two days ahead in the future that, "พรุ่งนี้" or "Prung Nee" (tomorrow) and "มะรืนนี้" or "Ma Reun Nee" (the day after tomorrow), respectively.
- We also have a word that haven't used widely in Thaliand. That is "มะเรื่องนี้" or "Ma Reung Nee". It means the day after the day after tomorrow.

Counting Words

Past ← → Future

เมื่อวานซืน	เมื่อวาน	วันนี้	พรุ่งนี้	มะรืนนี้	มะเรื่องนี้
Meu Wan Suen	Meu Wan	Wan nee	Prung Nee	Ma Ruan Nee	Ma Reung Nee
	Yesterday	Today	Tomorrow		

Exponential Notation

$10 \times 10 \downarrow 10^2$ (Base, Exponent) $\qquad 10 \times 10 \times 10 \downarrow 10^3 \qquad 10 \times 10 \times 10 \times 10 \downarrow 10^4$

10^1 is read "ten to the first power"
10^2 is read "ten to the second power" or "ten squared"
10^3 is read "ten to the third power" or "ten cubed"
10^4 is read "ten to the forth power"

10^a can be also read "ten to the power a"

Exponential Notation

Example 1 Evaluate 10^4.

Solution $10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$

Check Point 1 Evaluate 10^5 and 10^6.

Exponential Notation

Example 2 Evaluate the following: a) 8^2 b) 6^3 c) 2^5 d) 18^1

Solution
a) $8^2 = 8 \times 8 = 64$
b) $6^3 = 6 \times 6 \times 6 = 216$
c) $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$
d) $18^1 = 18$

Check Point 2 Evaluate the following: a) 7^2 b) 5^3 c) 2^6 d) 9^1

Hindu-Arabic Numeral System

- An important characteristic of our Hindu-Arabic system is that we can write the numeral system for any number using only ten symbols. The ten symbols are

0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

These symbols are called digit.
"Digit" is from the Latin word for fingers.

Hindu-Arabic Numeral System

The place value of the first digit on the right is 1 or 10^0.

The place value of the second digit from the right is 10 or 10^1.

The place value of the third digit from the right is 100 or 10^2 …

Hindu-Arabic Numeral System

The positional value in the system are based on powers of ten and are

$$\ldots, 10^5, 10^4, 10^3, 10^2, 10^1, 1.$$

For example, we can write 653 in expanded form as the following

$$653 = (6 \text{ x } 100) + (5 \text{ x } 10) + (3 \text{ x } 1)$$
$$= (6 \text{ x } 10^2) + (5 \text{ x } 10^1) + (3 \text{ x } 1)$$

Hindu-Arabic Numeral System

Example 3 Write the following numbers in expanded form:

a) 3,407 b) 53,525

Solution

a) $3{,}407 = (3 \text{ x } 1000) + (4 \text{ x } 100) + (0 \text{ x } 10) + (7 \text{ x } 1)$
$= (3 \text{ x } 1000) + (4 \text{ x } 100) + (7 \text{ x } 1)$
$= (3 \text{ x } 10^3) + (4 \text{ x } 10^2) + (7 \text{ x } 1)$

Hindu-Arabic Numeral System

b) 53,525

$= (5x10,000)+(3x1000)+(5x100)+(2x10)+(5x1)$

$= (5x10^4)+(3x10^3)+(5x10^2)+(2x10)+(5x1)$

Check Point 3 Write the following numbers in expanded form:

a) 12,067 b) 975,301

Hindu-Arabic Numeral System

Example 4 Express each expanded form as a Hindu-Arabic numeral:

a) $(7 \text{ x } 10^3) + (5 \text{ x } 10^1) + (4 \text{ x } 1)$

b) $(6 \text{ x } 10^5) + (8 \text{ x } 10^1)$

Solution a) $(7\text{x } 10^3) + (5 \text{ x } 10^1) + (4 \text{ x } 1)$

$= (7x10^3) + (0x10^2) + (5x10^1) + (4x1)$

$= 7,054$

Hindu-Arabic Numeral System

b) $(6\text{x } 10^5) + (8 \text{ x } 10^1)$

$= (6x10^5) + (0x10^4) + (0x10^3) + (0x10^2)$

$+ (8x10^1) + (0x1)$

$= 600,080$

Hindu-Arabic Numeral System

Check Point 4 Express each expanded form as a Hindu-Arabic numeral:

a) $(6 \text{ x } 10^3) + (7 \text{ x } 10^1) + (3 \text{ x } 1)$

b) $(8 \text{ x } 10^4) + (9 \text{ x } 10^2)$

Hindu-Arabic Numeral System

- Example 3 and 4 show that 0 is very important in Hindu-Arabic system. 0 was invented for nothingness. Moreover, we use 0 as a placeholder. So, we can see that 3407 and 347 are different. This idea changes the way we think about the world.

Early Positional Systems

- Hindu-Arabic system developed for many centuries. We found digits carved on ancient Hindu pillars over 2,200 years old.
- The Italian mathematician Leonardo Fibonacci (1770-1250) brought this system to Europe in 1202.
- Hindu-Arabic was into widespread use after printing was invented in 15th Century.

Early Positional Systems

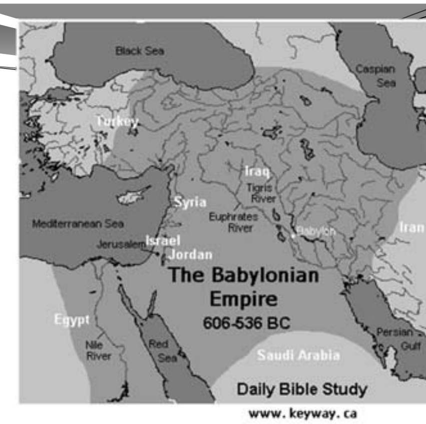
We can also use positional systems with powers of any number, not just 10. For example, for our system of time, we use powers of 60:

1 minute = 60 seconds

1 hour = 60 minutes = 60 x 60 seconds = 60^2 seconds

	Hour	-	Minute	-	Second
(Seconds)	60^2		60		1

The Babylonian Numeration System



Source: http://www.keyway.ca

- Balylon was the center of Babylonian civilization that lasted for about 1,400 years between 2000 B.C. and 600 B.C. The city is 55 miles south from present-day Baghdad.
- The Babylonians wrote on wet clay.

The Babylonian Numeration System

- The place values in the Babylonian system use power of 60. The place value are

$$\ldots,\ 60^3,\ 60^2\quad 60^1\quad 1$$

$60^2 = 60 \text{ x } 60 = 3600$

$60^3 = 60 \text{ x } 60 \text{ x } 60 = 216{,}000$

The Babylonian Numeration System

- Numbers 1-60 can be written with two symbols. That is easy for recording on wet clay. We read it from left to right as present Hindu-Arabic system.

Babylonian numerals	∨	<
Hindu-Arabic numerals	1	10

- We found that there is no symbol for 0 in the Babylonian system.
- The Babylonians left a space to distinguish the various place values.

The Babylonian Numeration System

- Babylonian civilization is still in our present culture in many ways, such as time measurement: 60 seconds is 1 minute and 60 minutes is 1 hour.
- Babylonians used mathematics in astronomy as well. They used a number to show degrees of a circle, that is 360 degrees. 360 is a multiple of 60.
- It is interesting to notice that 60 can be divided easily since it can be divided by many positive numbers including 1, 2, 3, 4, 5, 6, 10, 12, 15, 30 and 60.

The Babylonian Numeration System

∨   <   ∨ ∨

$(1 \text{ x } 60^2) + (10 \text{ x } 60^1) + (2 \text{ x } 1)$

$= (1 \text{ x } 3600) + (10 \text{ x } 60) + (2 \text{ x } 1)$

$= 3600 + 600 + 2$

$= 4{,}202$

The Babylonian Numeration System

Example 5 Write ∨∨ <∨ <<∨∨ as Hindu-Arabic numeral.

Solution

∨∨ <∨ <<∨∨

$(2 \text{ x } 60^2) + (11 \text{ x } 60^1) + (22 \text{ x } 1)$

$= (2 \text{ x } 3600) + (660) + 22$

$= 7200 + 660 + 22 \quad = 7{,}882$

The Babylonian Numeration System

Check Point 5 Write ∨∨∨ << <<<∨ as Hindu-Arabic numeral.

The Babylonian Numeration System

- Since there is on 0 in the Babylonian numeral system, Babylonians will leave a blank space as 0 used in our present Hindu-Arabic system.
- However, that could make us confused to know where they putted space, especially space on the right hand side.
- Later Babylonians decided to developed a symbol for that blank space.

Mayan Numeration System

source: iasanalysis.wordpress.com

- The Maya is a tribe of Central American Indians.
- Its peak is between 300-1000 A.D. Their civilization covered many area of present countries including parts of Mexico, all of Belize and Guatemala, and part of Hondurus.

Mayan Numeration System

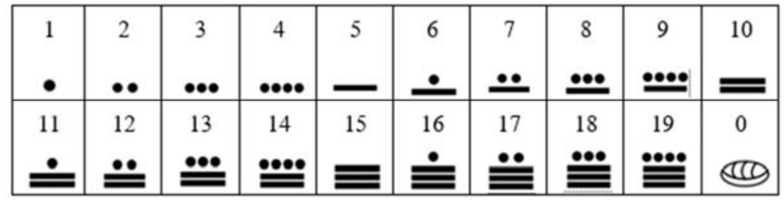
El Castillo. Chichen Itza, Yucatan, Mexico
www.charismanews.com

Aztec Calendar (Adapted from Mayan Calendar)
www.webexhibits.org

- They were famous on magnificent architecture, astronomical and mathematical knowledge, and excellent in arts.
- They used a symbol for zero in their numeral system before other systems.

Mayan Numeration System

Mayan Numerals



Mayan Numeration System

- The place values in the Mayan system are

…, $18\text{x } 20^3$, $18 \text{ x } 20^2$, $18 \text{ x } 20$, 20 , 1.

$18 \text{ x } 20 = 360$

$18 \text{ x } 20 \text{ x } 20 = 7{,}200$

$18 \text{ x } 20 \text{ x } 20 \text{ x } 20 = 144{,}000$

Mayan Numeration System

- We noticed that Mayans used 18x20 instead of 20^2. The reason for this might be that in their system one year is 360 days.
- Mayan numeral system is expressed vertically. The place value at the bottom of the column is 1.
- **Example 6** Write  as a Hindu-Arabic numeral.

Mayan Numeration System

Solution

Mayan Numeral		Hindu-Arabic Numeral		Place Value		
	=	14	x	7,200	=	100,800
	=	0	x	360	=	0
	=	7	x	20	=	140
	=	12	x	1	=	12
						100,952

Mayan Numeration System

- **Check Point 6** Write  as a Hindu-Arabic numeral.

1.1.2 Number Bases in Positional Systems

- It seems reasonable that we use base ten numberals because we have 10 fingers or 10 toes. However, when we look at history of many cultures, we find that people used numberals with bases 2, 3, 4, 5, 12, 20 and 60.
- When we create 3D photos, look at online information, and edit a photo in a computer, computers don't use base ten but use base two, consisting only two numbers 0 and 1.

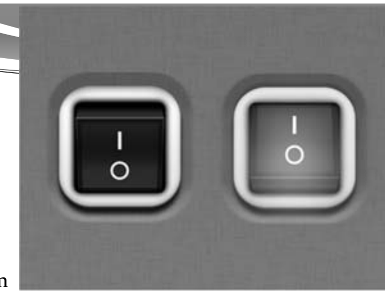
1.1.2 Number Bases in Positional Systems

- We will study numberals in many bases and we will then appreciate with the nature of positional systems. We will also understand more about the calculation we are using in everyday life. Moreover, we will know how to see the world with the computer point of view.

Base Ten system

- Symbols of Hindu-Arabic numbers we use in the present were used for the first time by Indian mathematicians in Brahmin Gupta era. Then they had been taken to the Arabian territory. Later, tourists, merchants, and conquerors in many lands published this numbers to North Africa and the Iberian Peninsula. In the 12th century, they were expanded into Europe.
- A book making people know Hindu-Arabic numbers widely is **Liber Abaci**. It is a calculation book written by Leonardo Fibonacci of Pisa and published in 1202.

Base Ten system



ux.stackexchange.com

- Computers and a lot of modern technology use “switches” that are in the "off" or "on" state. The binary numbers are used instead of "off" and "on" status with 0 and 1, respectively.
- Because we do not think binary. So we need to show how to convert numbers between binary and decimal.

Changing Numerals in Bases Other Than Ten to Base Ten

- The place values in base two system are

$$\ldots, \quad 2^4, \quad 2^3, \quad 2^2, \quad 2^1, \quad 1$$

$$\ldots, \quad 16, \quad 8, \quad 4, \quad 2, \quad 1$$

- $$\begin{aligned} 1011_{two} &= (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 1) \\ &= (1 \times 8) + (0 \times 4) + (1 \times 2) + (1 \times 1) \\ &= 8 + 2 + 1 \\ &= 11 \end{aligned}$$

Changing Numerals in Bases Other Than Ten to Base Ten

- Base Ten 0 1 2 3 4 5 6 7 8 9
- Base Two 0 1
- Base Ten 1 2 3 4 5 6 7 8 9 10
- Base Two 1_{two} 10_{two} 11_{two} 100_{two} 101_{two} 110_{two} 111_{two} 1000_{two} 1001_{two} 1010_{two}
- Base b system : The place values are

$$\ldots, \quad b^4, \quad b^3, \quad b^2, \quad b^1, \quad 1$$

The digit symbols are $0, 1, 2, 3, 4, 5, \ldots, b-1$

Changing Numerals in Bases Other Than Ten to Base Ten

Base	Digit Symbols	Place Values
2	0 , 1	$\ldots, 2^3, 2^2, 2^1, 1$
3	0 , 1 , 2	$\ldots, 3^3, 3^2, 3^1, 1$
4	0 , 1 , 2 , 3	$\ldots, 4^3, 4^2, 4^1, 1$
.		
.		
10	0 , 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9	$\ldots, 10^3, 10^2, 10^1, 1$

How to Change to Base Ten

1. Find the place value for each digit in the numeral.
2. Multiply each digit in the numeral by its respective place value.
3. Find the sum of the products in 2.

How to Change to Base Ten

Example 1 Convert 100101_{two} to base ten.

Solution Place values are …, 2^5, 2^4, 2^3, 2^2, 2^1, 1.

100101_{two}

$= (1x2^5)+(0x2^4)+(0x2^3)+(1x2^2)+(0x2^1)+(1x1)$

$= 32 + 4 + 1$

$= 37$

- **Check Point** 1 Convert 110011_{two} to base ten.

Sixteen Base System

- 0 1 2 3 4 5 6 7 8 9 A B C D E F

$A \to 10$, $B \to 11$, $C \to 12$, $D \to 13$, $E \to 14$, $F \to 15$

Sixteen Base System

- **Example 2** Convert $EC7_{sixteen}$ to base ten.

Solution Place values are

$$\ldots, 16^2, 16^1, 1.$$

E is 14 in base ten.

C is 12 in base ten.

$$EC7_{\text{sixteen}} = (14 \times 16^2) + (12 \times 16^1) + (7 \times 1)$$
$$= 3{,}584 + 192 + 7$$
$$= 3{,}783$$

Sixteen Base System

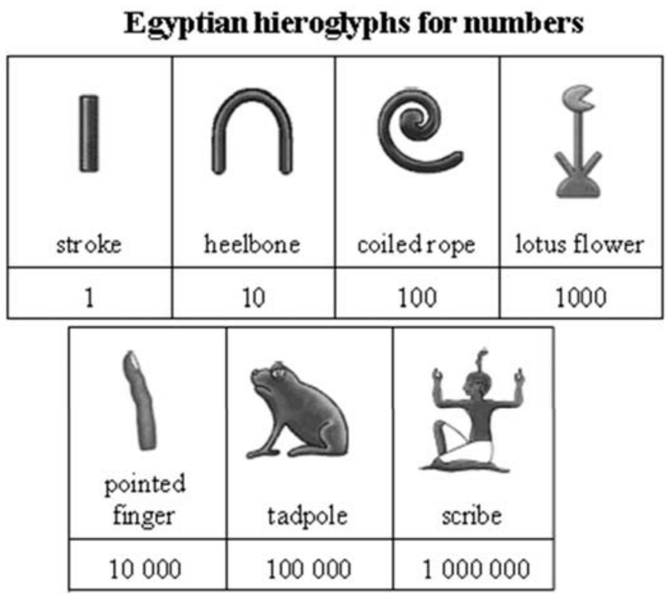
Check Point 2 Convert $AD4_{sixteen}$ to base ten.

1.1.3 Early Numeration System

- We have already seen that the Hindu-Arabic system has been very successful in its implementation because it can write a number by using only ten symbols. Moreover, the calculation in this system is quite easy.
- If we look back to some early numeration system, such as Roman and Egyptian numerals, we can clearly see that the Hindu-Arabic system we use today is more prominent than any other system in the past.

Egyptian Numeration System

Ancient Egyptians used several numeral systems. The oldest system was developed about 5,400 years ago, as shown in the table below. It is noteworthy that each number is in the form of power of ten.



Source: nicolelvanloon.wordpress.com/2015/02/04/grouping-and-number-bases

Egyptian Numeration System

- For most numbers, writing in the Egyptian system is longer than writing in the Hindu-Arabic system we use today because Egyptian numerals use duplicate symbols. If we want to write 543, we will write as
  100 100 100 100 100 10 10 10 10 1 1 1.

When we write as an Egyptian numeral, it will be

Roman Numeral System

The Roman numeral system was developed between 500 BC and AD 100 for taxation and trading in the vast Roman Empire. Roman numerals can be shown in the table below.

Roman Numeral	**I**	V	**X**	L	**C**	D	M
Hindu-Arabic Numeral	1	5	10	50	100	500	1000

Roman Numeral System

A way to remembering Roman Numerals can be done by writing smaller to bigger numerals as the following English sentence:

"If Val's X-ray Looks Clear, Don't Medicine."

Val is a nickname for many names in English, including Valdemar, Valentin, Valentino, Valery and Valley.

Roman Numeration System

We also see the use of Roman numerals today, such as some watch faces and the inscriptions on the buildings of the West, built in both past and present.

The left image is a wrist watch using Roman numerals. The right image shows the Roman inscription on the concrete wall of the building, which says "Administration Panama Canal A.D. MDCCCCXIV", stating that the building was built in 1914.

Roman Numeration System

- In Roman numeration system, if the symbols are lowered from left to right, we add each number together. For example, MD = 1000 + 500 = 1500. However, if the symbols are increased from left to right, we substract the left number from the right number. For example, IX = 10-1 = 9 and CM = 1000-100 = 900.
- In the past, there was no fixed rule. So some numbers can be written in several ways. A few hundred years ago, the rules for writing were clearly defined as shown below.

Roman Numeral System

1. Hindu-Arabic numerals can be broken down into Roman numerals. For example, 1957 consists of 1, 9, 5, and 7. For writing a Roman numeral, we should separate each number from others. Since M = 1000, CM = 900, L = 50, XII = 7, 1957 = MCMLXII.

2. For each symbol of I, X, C and M, we can write at most 3 repeated consecutive symbols. However, each symbol of D, L and V are not allowed to rewritten next to itself. For example, 4 is IV not IIII and 100 is C not DD.

Roman Numeral System

3. There are only 3 numerals we can subtract from other numerals. Those are I (1), X (10) and C (100).

I can be subtracted from **V** and **X** only.

X can be subtracted from **L** and **C** only.

C can be subtracted from **D** and **M** only.

Therefore, for subtraction, the numerals on the right hand side must be from one of the next two larger numerals. With this rule, XC can be written for 90 but IC can not be written because C are not one of two larger numerals from I.

Roman Numeration System

4. There is only one symbol that can be subtracted from a larger number. So we do not write 8 as IIX, but we write VIII only.

Although the Romans used the decimal system to count integers, for fractional systems they used base twelve system. The reason is that 12 can be divided by 2, 3, 4, and 6. This can make management easier.

The Last Time I Saw Paris is a movie created by the film company MGM in 1954. The company used a Roman numeral "MCMXLIV" showing their copyright. That means that the copyright has started in 1944 or 10 years before showing the movie. That means the copyright of this movie was reduced from 28 years to 18 years. The company did not make any modifications because it was thought that the copyright period was long enough.

Roman Numeration System

Example 1 Write CLXXXI as a Hindu-Arabic numeral.

Solution

C L XXX I

$$= 100 + 50 + (10+10+10) + 1 = 181$$

- Check Point 1 Write MMCCLXIII as a Hindu-Arabic numeral.

Roman Numeration System

Example 2 Write MCMXCIV as a Hindu-Arabic numeral.

Solution

M CM XC IV

$$= 1000 + (1000-100) + (100-10) + (5-1)$$

$$= 1000+900+90+4 = 1994$$

- Check Point 2 Write MCDXLIX as a Hindu-Arabic numeral.

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